

Wave-Particle Duality and the Hamilton-Jacobi Equation

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Abstract

The Hamilton-Jacobi equation of relativistic quantum mechanics is revisited. The equation is shown to permit solutions in the form of breathers (oscillating/spinning solitons), displaying simultaneous particle-like and wave-like behavior. The de Broglie wave thus acquires a clear *deterministic* meaning of a wave-like excitation of the classical action function.

The problem of quantization in terms of the breathing action function and the double-slit experiment are discussed.

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1. Introduction

A mathematical representation of the dual wave particle nature of matter remains one of the major challenges of quantum theory [1-7]. The present study is an attempt to resolve this issue through an appropriately revised Hamilton-Jacobi formalism.

Consider the relativistic Hamilton-Jacobi (HJ) equation for a particle in an electromagnetic field,

$$(1/c^2) (\partial S/\partial t + eU)^2 - (\nabla S - e\mathbf{A}/c)^2 = m^2 c^2 \quad (1)$$

Here U, \mathbf{A} are scalar and vector potentials of the field obeying the Lorentz calibration condition,

$$(1/c) \partial U/\partial t + \text{div} \mathbf{A} = 0 \quad (2)$$

The trajectory $\mathbf{x}(t)$ of the particle is governed by the equation [1],

$$\frac{d\mathbf{x}}{dt} = -\frac{c^2 (\nabla S^{(0)} - e\mathbf{A}/c)}{\partial S^{(0)}/\partial t + eU} \quad , \quad (3)$$

where $S^{(0)}$ is an appropriate action function of the system. As is well known, the trajectories $\mathbf{x}(t)$ are *characteristics* of Eq. (1) [8], which are, in turn, the *traces* of small perturbations. Indeed, let's represent the action function as

$$S = S^{(0)} + s \quad (4)$$

where s is a perturbation. Assuming s to be small, Eq. (1) yields the linear equation,

$$(1/c^2) (\partial S^{(0)}/\partial t + eU) \partial s/\partial t - (\nabla S^{(0)} - e\mathbf{A}/c) \nabla s = 0 \quad , \quad (5)$$

whose characteristics are identical to those of Eq. (1).

Let

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{c} \quad (6)$$

be a set of independent integrals of Eq. (3), \mathbf{c} being a constant vector. Then the general solution of Eq. (5) may be written as,

$$s = s[\mathbf{f}(\mathbf{x}, t)] \quad (7)$$

The perturbation s is, therefore, advected along the trajectory $\mathbf{x}(t)$ with the velocity $\mathbf{v} = d\mathbf{x}/dt$. If the perturbation is localized enough it will mimic the motion of the particle. As an example, consider the case of a free particle ($U = 0, \mathbf{A} = 0$), where

$$S^{(0)} = -Et + \mathbf{p} \cdot \mathbf{x} \quad (E^2/c^2 = p^2 + m^2c^2) \quad (8)$$

Eq. (5) then yields,

$$s = s(\mathbf{x} - \mathbf{v}t), \quad (9)$$

where

$$\mathbf{v} = c^2 \mathbf{p} / E \quad (10)$$

In the linear approximation the perturbation is advected without changing its shape. However, in a nonlinear description, due to the Huygens principle, the perturbation will gradually decay thereby implying stability (albeit nonlinear) of the regular solution $S^{(0)}$.

The question is whether it is possible to modify the HJ equation (1) so that the new equation would allow for localized, nonspreading and nondecaying perturbations (excitations) of the regular action function. Moreover, if the localized excitation breathes (oscillates/spins), one would end up with a *deterministic* model for a particle with quantum-like features. As we intend to show, this kind of behavior can be successfully modeled by the conventional quantum Hamilton-Jacobi (QHJ) equation,

$$(1/c^2) (\partial S / \partial t + eU)^2 - (\nabla S - e\mathbf{A}/c)^2 = m^2c^2 + i\hbar \square S \quad , \quad (11)$$

whose capacity, it transpires, has simply not been fully explored.

2. A free particle

As is well known, the QHJ equation (11) is a transformed version of the linear Klein-Gordon (KG) equation

$$(1/c^2) (\partial/\partial t + ieU/\hbar)^2 \Psi - (\nabla - ie\mathbf{A}/c\hbar)^2 \Psi + (mc/\hbar)^2 \Psi = 0, \quad (12)$$

obtained through the substitution,

$$\Psi = \exp(iS/\hbar) \quad (13)$$

Consider first the case of a free particle ($U = 0, \mathbf{A} = 0$) where Eq. (12) becomes

$$\square \Psi + (mc/\hbar)^2 \Psi = 0 \quad (\square = (1/c^2) \partial^2/\partial t^2 - \nabla^2) \quad (14)$$

The KG equation (14) allows for a two-term spherically symmetric solution

$$\Psi = \exp[-i(mc^2/\hbar)t] + \alpha \exp(-i\omega t) j_0(kr) \quad (15)$$

where

$$\omega = c\sqrt{k^2 + (mc/\hbar)^2}, \quad (16)$$

$r = \sqrt{x^2 + y^2 + z^2}$, α is a free parameter, and

$$j_0(kr) = \sin(kr)/kr \quad (17)$$

is the zeroth-order spherical Bessel function.

The second term in Eq. (15) is a standing spherically symmetric breather, $|\alpha|$ being its intensity.

In terms of the action function S , by virtue of (13), Eq. (15) readily yields

$$S = mc^2 t - i\hbar \ln \{1 + \alpha \exp[-i(\omega - mc^2/\hbar)t] j_0(kr)\} \quad (18)$$

Here the first term corresponds to the classical action function, $S^{(0)} = -mc^2t$, for a free particle in the rest system while the second term represents its localized excitation, oscillating and *nonspreading*.

Let's set the frequency of oscillations in Eq. (18) in accordance with the de Broglie postulate that each particle at rest can be linked to an internal 'clock' of frequency mc^2/\hbar . The frequency ω in Eq. (15) should therefore be specified as

$$\omega = 2(mc^2/\hbar) \quad (19)$$

Hence, by virtue of Eq. (16),

$$k = \sqrt{3}(mc/\hbar) \quad (20)$$

Eq. (18) thus becomes,

$$S = -mc^2t - i\hbar \ln \left\{ 1 + \alpha \exp \left[-i \left(\frac{mc^2}{\hbar} \right) t \right] j_0 \left[\sqrt{3} \left(\frac{mc}{\hbar} \right) r \right] \right\} \quad (21)$$

Note that in Eq. (21) the frequency is not affected by the nonlinearity of the system, preserving its value irrespective of the breather intensity.

Away from the breather's core ($r \gg \hbar/mc$),

$$S = -mc^2t - i\alpha\hbar \exp \left[-i \left(\frac{mc^2}{\hbar} \right) t \right] j_0 \left[\sqrt{3} \left(\frac{mc}{\hbar} \right) r \right] \quad (22)$$

The oscillations are therefore asymptotically *monochromatic*, again in accord with the de Broglie picture [1].

Similar to oscillations of an ideal pendulum, the breather (21) is stable to small perturbations. The stability follows from the linearity of the KG equation. Due to the linearity there is no coupling between the basic solution (15) and its perturbation, which also obeys the KG equation. Therefore, if the initial perturbation is small it will remain so indefinitely. The stability here is

understood in a weak (non-asymptotic) sense.

Physically relevant action functions are quite specific *global* solutions defined over the entire time axis $-\infty < t < \infty$. Moreover, they may be multiple valued and bound in space. Such solutions cannot be obtained through a conventional initial-value problem unless suitable initial conditions are known in advance.

Until now we have dealt with a particle at rest. For a particle moving at a constant velocity v along, say, x - axis, the corresponding expression for the action function is readily obtained from Eq. (21) through the Lorentz transformation,

$$t \rightarrow \frac{t - xv/c^2}{\sqrt{1 - (v/c)^2}}, \quad x \rightarrow \frac{x - vt}{\sqrt{1 - (v/c)^2}} \quad (23)$$

The transformed Bessel function j_0 will then mimic the motion of the classical particle while the transformed temporal factor $\exp[-i(mc^2/\hbar)t]$ will turn into the associated de Broglie wave, thereby demonstrating *simultaneous* particle-like and wave-like behavior. Moreover, unlike conventional quantum mechanics, here the modulated de Broglie wave acquires the clear *deterministic* meaning of a wave-like excitation of the action function, a complex-valued potential in configuration space. If, as is conventional, we associate the gradients $-\partial S/\partial t, \nabla S$ with the particle energy E and momentum \mathbf{p} , then the Einstein relation $(1/c^2)E^2 = p^2 + m^2c^2$ appears to hold only far from the \hbar/mc - wide breather's core, or *on average* over the entire breather. The correspondence with classical relativistic mechanics is therefore complied with.

In addition to spherically symmetric breathers, Eq. (14) also permits asymmetric breathers, spinning around some axis. In the latter case the second term of Eq.(15) should be replaced by

$$\alpha \exp \left[-2i \left(\frac{mc^2}{\hbar} \right) t + in\phi \right] j_l \left[\sqrt{3} \left(\frac{mc}{\hbar} \right) r \right] P_l^n (\cos \theta), \quad (24)$$

where j_l, P_l^n are high-order spherical Bessel functions and associated Legendre functions. It would be interesting to ascertain in what way (if any) the double-valued spin- $\frac{1}{2}$ breather may be linked

to the Dirac wave function.

The next question is how to reproduce quantization directly in terms of the breathing action function. The geometrically simplest situation, where such an effect manifests itself, is the periodic motion of an otherwise free particle over a closed interval $0 < x < d$. In this case the field-free version of Eq.(11) must be considered jointly with two boundary conditions,

$$\begin{aligned}\partial S(0, y, z, t)/\partial t &= \partial S(d, y, z, t)/\partial t, \\ \partial S(0, y, z, t)/\partial x &= \partial S(d, y, z, t)/\partial x\end{aligned}\tag{25}$$

Any classical action function for a free particle,

$$S = -Et + px\tag{26}$$

is clearly a solution of this problem. However, in the case of a breathing action function the situation proves to be different. Thanks to the boundary conditions (25), the moving breather interacts with itself, and this may well lead to its self-destruction unless some particular conditions are met.

Consider first the simplest case of a particle at rest ($v = 0$). The pertinent solution is readily obtained by converting the problem for a finite interval into a problem for an infinite interval ($-\infty < x < \infty$) filled with a d -periodic train of standing breathers, assumed to be spherical for simplicity. The resulting action function then reads,

$$S = -mc^2t - i\hbar \ln \left\{ 1 + \alpha \exp \left[-i \left(\frac{mc^2}{\hbar} \right) t \right] \sum_k (j_0)_{d,0}^{(k)} \right\},\tag{27}$$

where

$$(j_0)_{d,0}^{(k)} = \frac{\sin \left[\sqrt{3} (mc/\hbar) r_{d,0}^{(k)} \right]}{\sqrt{3} (mc/\hbar) r_{d,0}^{(k)}},\tag{28}$$

$$r_{d,0}^{(k)} = \sqrt{(x - kd)^2 + y^2 + z^2} \quad (k = 0, \pm 1, \pm 2, \dots) \quad (29)$$

Here the second subscript stands for $v = 0$.

The action function for a moving particle ($v \neq 0$) is obtained from (27) (28) (29) through the Lorentz transformation (23), provided d is replaced by $d/\sqrt{1 - (v/c)^2}$. The latter step is needed to balance the relativistic contraction, and thereby to preserve the spatial period (d) of the system.

The resulting action-function thus becomes,

$$S = -Et + px - i\hbar \ln \left\{ 1 + \alpha \exp \left[i \left(\frac{-Et + px}{\hbar} \right) \right] \sum_k (j_0)_{d,v}^{(k)} \right\}, \quad (30)$$

where

$$(j_0)_{d,v}^{(k)} = \frac{\sin \left[\sqrt{3} (mc/\hbar) r_{d,v}^{(k)} \right]}{\sqrt{3} (mc/\hbar) r_{d,v}^{(k)}}, \quad (31)$$

$$r_{d,v}^{(k)} = \sqrt{\left(\frac{x - vt - kd}{\sqrt{1 - (v/c)^2}} \right)^2 + y^2 + z^2} \quad (32)$$

The spatial $2\pi\hbar/p$ - periodicity of $\exp [i (-Et + px) / \hbar]$ is compatible with the spatial d - periodicity of $\sum_k (j_0)_{d,v}^{(k)}$ only if

$$dp = 2\pi n\hbar \quad (n = 0, 1, 2, 3, \dots), \quad (33)$$

which recovers the familiar Bohr-Sommerfeld quantum condition. While the particle velocity is clearly subluminal its communication with boundary conditions is *superluminal*, which does not violate the Lorenz-invariance of the system.

The above solution (30)-(33) may be easily adapted for the problem of a particle shuttling between two perfectly reflecting walls, $x = 0$ and $x = d/2$. To handle the double-valuedness of the pertinent action function the trajectory of the particle, following the Einstein-Keller topological approach [9,10], should be placed on the double-sheeted strip, $0 < x < d/2$, $-\infty < y < \infty$, $z = \pm 0$.

Thereupon the problem reduces to the previous one.

3. A particle in a slowly varying field

The de Broglie postulate holds at least for breathers exposed to slowly varying potentials, characterized by spatio-temporal scales much larger than \hbar/mc , \hbar/mc^2 . Indeed, as may be readily shown, for slowly varying potentials, Eq. (21) becomes

$$S = -(mc^2 + eU)t + \frac{e}{c}\mathbf{A}\cdot\mathbf{x} - i\hbar \ln \left\{ 1 + \alpha \exp \left[-i \left(\frac{mc^2}{\hbar} \right) t \right] j_0 \left[\sqrt{3} \left(\frac{mc}{\hbar} \right) r \right] \right\} \quad (34)$$

So, unlike the action function as a whole, the frequency of its oscillations in the rest system is not affected by the field. This invariance is untenable for the wave function Ψ (13), which therefore cannot serve as a physically objective representation of the de Broglie clock.

For a particle *moving* in a slowly varying field the ‘fast’ spatio-temporal coordinates \mathbf{x}, t in Eq. (34) should be subjected to the Lorentz transformation, with the velocity \mathbf{v} regarded as a slowly varying vector.

The action function (34) and its Lorentz transformed version pertain to the interior of the breather (inner solution). Away from the breather’s core the action function is described by the regular (breather-free) solution of the QJH equation $S^{(0)}$, involving only large spatio-temporal scales (outer solution). The inner solution is clearly affected by the outer solution through the velocity field \mathbf{v} , while the reverse influence does not take place, at least not for the leading order asymptotics. The simplest picture emerges in the nonrelativistic *semiclassical* limit [11] where the uniformly valid asymptotic solution may be represented as,

$$S = -mc^2 t + S_{sc} - i\hbar \ln \left\{ 1 + \alpha \exp \left[-i \left(\frac{mc^2 + mv^2/2}{\hbar} \right) t \right] \exp \left[i \left(\frac{\mathbf{p}\cdot\mathbf{x}}{\hbar} \right) \right] j_0 \left[\sqrt{3} \left(\frac{mc}{\hbar} \right) r \right] \right\} \quad (35)$$

Here S_{sc} is the semiclassical action function governed by the equation

$$\frac{\partial S_{sc}}{\partial t} + \frac{1}{2m} \left(\nabla S_{sc} - \frac{e}{c}\mathbf{A} \right)^2 + eU = -\frac{i\hbar}{m} \nabla^2 S_{sc} \quad , \quad (36)$$

where S_c is the classical action function obeying the equation

$$\frac{\partial S_c}{\partial t} + \frac{1}{2m} \left(\nabla S_c - \frac{e}{c} \mathbf{A} \right)^2 + eU = 0 \quad (37)$$

The argument r in $j_0 \left[\sqrt{3} (mc/\hbar) r \right]$ is defined as

$$r = |\mathbf{x} - \mathbf{x}_p(t)| \quad , \quad (38)$$

where $\mathbf{x}_p(t)$ is the classical trajectory of the particle described by the equation

$$\frac{d\mathbf{x}_p}{dt} = \frac{1}{m} \mathbf{p} - \frac{e}{c} \mathbf{A} \quad (\mathbf{p} = \nabla S_c) \quad (39)$$

One therefore may readily see the connection between the new formalism and those of Schrödinger, Bohr and Sommerfeld. The Schrödinger formalism pertains to the outer solution which, under appropriate conditions, provides the data on the particle's range of energies, but says nothing about its trajectory. The information about the particle's trajectory comes from the inner solution, connecting de Broglie waves with the Bohr-Sommerfeld theory. Recall that for bound systems in the semiclassical limit (high quantum numbers), the Schrödinger and Bohr-Sommerfeld formalisms lead to identical energy spectra.

4. The effect of boundaries

The outlined formalism seems fully compatible with the double-slit experiment. Diffraction pictures obtained from electron beams of very low intensity [1,4] provide convincing evidence that the double-slit experiment is actually a one-particle effect where the particle communicates with distant boundaries that affect its trajectory. The breather passing through a slit 'feels' whether the other slit is open or closed, and changes its trajectory accordingly. A geometrically simpler system, where one may readily observe the impact of boundaries, is a breather moving in a uniform *circular* motion through a thin toroidal duct, e.g. doughnut shaped carbon nanotube. In classical

mechanics such a motion would be impossible without external forcing. In quantum mechanics however, the bending of the trajectory is a manifestation of the rotating nature of the pertinent KG solution. The problem may be easily tackled analytically in the intermediate limit when the width of the duct d is comparable to the de Broglie wavelength, small compared to the torus centerline radius R , and large compared to the breather's width \hbar/mc .

For the leading order asymptotics the resulting action function (inner solution), written in cylindrical co-ordinates ρ, ϕ, z , reads,

$$S = -Et + p_\phi R\phi - i\hbar \ln \left\{ 1 + \alpha \exp \left[i \left(\frac{-Et + p_\phi R\phi}{\hbar} \right) \right] j_0 \left[\sqrt{3} \left(\frac{mc}{\hbar} \right) r \right] \right\}, \quad (40)$$

where

$$r = \sqrt{(R\phi - v_\phi t)^2 + (\rho - R)^2 + z^2} \quad , \quad E = mc^2 + mv_\phi^2/2 \quad , \quad p_\phi = mv_\phi = n\hbar/R \quad (41)$$

Here v_ϕ, p_ϕ are the azimuthal velocity and momentum, $v_\phi \ll c$, $R\phi \sim |\rho - R| \sim \hbar/p_\phi$, $n\phi \sim 1$, and n is a large integer.

5. Concluding remarks

The proposed formalism is certainly related to de Broglie's double solution program [1]. Yet, unlike the latter, in the current model the breather is guided by a regular, generally nonwaving, action function $S^{(0)}$ rather than by a guiding wave solution (Sec 2). The guiding action function and its localized excitation (breather) are coupled through the *nonlinear* QHJ equation (11).

The de Broglie's double solution program is adjacent to the de Broglie-Bohm pilot-wave theory [1,4,7], which should be seen as a degenerate double solution theory, where the breather has been replaced by the particle position governed by an appropriate guidance equation. For unbound systems the pilot-wave theory and the current formulation are likely to correspond. However, for

bound systems the de Broglie-Bohm formalism, based on standing KG-Schrödinger waves, produce immobile particles [1,4], which contradicts the classical limit. In the present formulation, dealing with drifting nonspreading breathers and multiple valued action functions, this difficulty does not occur (Sec. 2).

At this stage it is difficult to see whether the amended QHJ formalism is indeed adequate enough to reproduce all the basic features of quantum-mechanical phenomenology. In any case, a few preliminary observations already show that a mathematical representation of unified wave-particle behavior is quite feasible, even within the framework of the conventional QHJ equation.

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